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Original Article

New 4-D Hyperchaotic System Derived from the Rikitake 3-D System:Properties and Application

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Abstract

A novel 4D hyperchaotic system with two positive Lyapunov exponents is created from a Rikitake 3D system using a feedback control strategy. Several dynamical properties are investigated both theoretically and numerically, including phase, equilibrium points, divergence, root stability, and Lyapunov exponents. Additionally, using MATLAB 2021, the numerical simulation validates the veracity of the theoretical conclusions. Finally, Multisim 14.2 was used to simulate and build the electrical circuit for a new system.

Keywords: Novel 4D, Hyperchaotic, Lyapunov exponents, Saddle-Focus, Dissipative

Introduction

Researchers have focused on four-term chaotic systems because they are algebraically easier than chaotic systems that have more terms. Over the past few decades, chaotic dynamical systems with different dimensions and their applications in many fields, such as encryption [1–3], chaos control [4, 5], and fuzzy logic [6, 7], electrical circuits [8, 9], ecology [10], neural networks [11], synchronizations [12], lasers [13]. Optimization algorithms, or genetic algorithms [14, 15] can also be used and combined with dynamical systems to create chaotic systems that can be used in specific applications, such as cryptography.

There are two types of attractors in chaotic systems: hidden and self-excited. Two positive Lyapunov exponents are present in several 4D systems, i.e., $(n-2)+ve\ LEs$ [16, 17], in contrast to other 3D systems that only have one positive Lyapunov exponent $(n-3)+ve\ LEs$ [18]. To enhance security, it is advisable to work with systems that have more than a positive Exponent of Lyapunov.

The following points describe the main contributions to this work:

– A new 4D hyperchaotic system with a Saddle-Focus equilibrium is derived from the Rikitake system.

– The proposed system is simple.

- This system has two positive Lyapunov exponents, i.e., satisfied (n-2)+ve LEs.
- The electronic circuit is performed via Multisim 14.2 software.

Rikitake System 3D

The following modelling approach for the Rikitake system was suggested [19]:

$$\begin{cases} \dot{x} = -ax + yz \\ \dot{y} = -ay + xz - bx \\ \dot{z} = 1 - xy \end{cases} \quad (1)$$

This system has seven terms, with $(a = 1, b = 1)$, and IC $= (0.8, 0.2, 0.4)$, The chaotic attractors in the system (1) have one positive exponent:

$$L_1 = 0.1283, \qquad L_2 = 0\,, \qquad L_3 = -2.1274,$$

and its Lyapunov dimension has $= (2.0603)$.

Four-dimensional systems creation

The paper discusses the introduction of a new model using a feedback control strategy. Specifically, it proposes modifying the second equation in the system (1) to create a novel 4D hyperchaotic system. This modification involves adding a fourth equation through the incorporation of a non-linear controller.

$$\begin{cases} \dot{x} = -ax + yz \\ \dot{y} = -ay + xz - bx - cw \\ \dot{z} = 1 - xy \\ \dot{w} = x - dy \end{cases} \quad (2)$$

where the system variables are x, y, z, w, and c, d are the control parameters, where $c \neq 0$. As shown in Fig.1, the described system (2) demonstrates hyperchaotic behavior under typical parameters:

$$(a, b, c, d) = (1.5, 10, 0.06, 1) \quad (3)$$

and specifying initial conditions (IC) for the system (2), as follows:

$$X(0) = (0.8, 0.4, 0.2, 0.7)^T \quad (4)$$

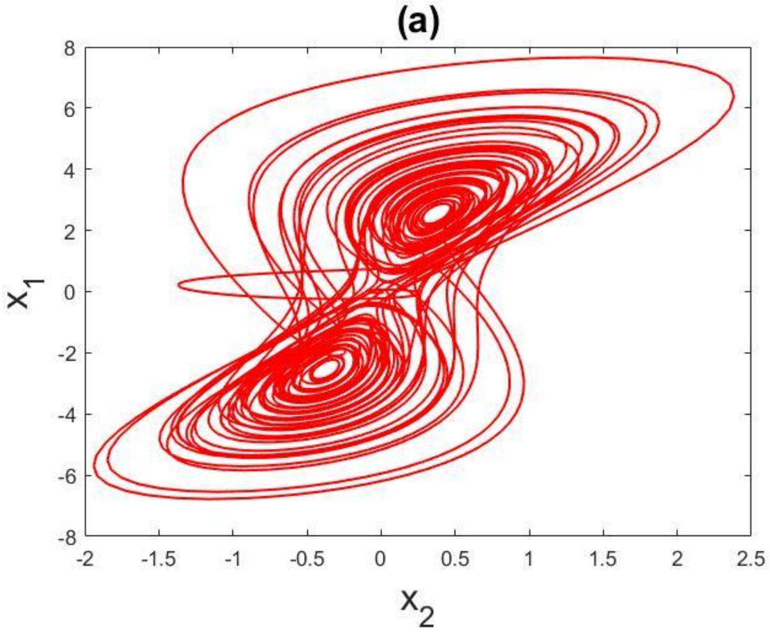

Figure 1. Hyperchaotic attractors of the system (2): $x_2 - x_1$ plane.

Properties dynamical

Equilibrium points

By solving the system's equations (2):

$$\begin{cases} -ax + yz = 0 \\ -ay + xz - bx - cw = 0 \\ 1 - xy = 0 \\ x - dy = 0 \end{cases} \quad (5)$$

The two equilibria points can be computed as follows:

$$\begin{cases} E_1\left(\sqrt{d}, \frac{1}{\sqrt{d}}, ad, \frac{-a+ad^2-bd}{c\sqrt{d}}\right) \\ E_2\left(-\sqrt{d}, \frac{-1}{\sqrt{d}}, ad, \frac{a-ad^2+bd}{c\sqrt{d}}\right) \end{cases} \quad (6)$$

Remark 1. There are two classifications under the new system (2).

– If $d \neq 0$, self-excited attractors,

– If $d = 0$, hidden attractors.

Remark 2. The computation of divergence for system (2) is as follows:

$$\nabla \cdot V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{w}}{\partial w} = -2a$$

Characteristic Equation and Roots and Stability

A system's Jacobian matrix (2) at point E_1:

$$J = \begin{bmatrix} -a & z & y & 0 \\ z - b & -a & x & -c \\ -y & -x & 0 & 0 \\ 1 & -d & 0 & 0 \end{bmatrix} \quad (7)$$

$$\Rightarrow J(E_1) = \begin{bmatrix} -a & ad & \frac{1}{\sqrt{d}} & 0 \\ ad-b & -a & \sqrt{d} & -c \\ -\frac{1}{\sqrt{d}} & -\sqrt{d} & 0 & 0 \\ 1 & -d & 0 & 0 \end{bmatrix} \qquad (8)$$

Applying the law $|\boldsymbol{J} - \boldsymbol{\lambda I}| = \mathbf{0}$, find the characteristic equation:

$$\lambda^4 + \underbrace{(2a)}_{A_1}\lambda^3 + \underbrace{\left(\frac{1}{d} + d - cd + abd + a^2 - a^2d^2\right)}_{A_2}\lambda^2 + \underbrace{\left(\frac{a}{d} + 3ad - b\right)}_{A_3}\lambda - \underbrace{2c}_{A_4} = 0 \qquad (9)$$

Theorem 1. System (2) has type Saddle-Focus equilibria points, and an unstable system.

Proof: Put the parameters (3) in equation (9), we get:

$$\lambda^4 + 2.6\,\lambda^3 + 14.94\,\lambda^2 - 4.8\,\lambda - 0.12 = 0$$

The eigenvalues:

$$\begin{cases} \lambda_1 = -0.0269 \\ \lambda_2 = 0.2520 \\ \lambda_{3,4} = -1.6125 \mp 3.8823i \end{cases}$$

Here λ_1 and λ_2 are real numbers with opposite signs. Due to this characteristic, the roots are categorized as Saddle-Focus, resulting in an unstable system. Consequently, the system is identified as a self-excited system.

Aim of the study

The study aims to develop a new four-dimensional hyperchaotic system derived from the Rikitake 3-D system, analyzes its dynamical properties through stability and Lyapunov exponent investigations. It also demonstrates its potential applications in secure communication and electronic circuit design.

System Analysis using Exponential and Lyapunov Dimension

In phase space, Lyapunov exponents represent the mean exponential rates of divergence or convergence and are the only tool available for measuring a dynamical system's behavior into chaotic and hyperchaotic [20]. Lyapunov exponents for a four-dimensional system (2) were proven using the parameters (3) and IC (4).

Based on the mathematical software Matlab 2021 and Wolf's algorithm [21], this system has two positive Lyapunov exponents, as illustrated in Fig. 2., besides the essential Lyapunov exponent

$$\begin{cases} LE_1 = 0.1538 \\ LE_2 = 0.0028 \\ LE_3 = \mathbf{-0.0005} \\ LE_4 = -3.1560 \end{cases} \Rightarrow \sum_{i=1}^{4} LE_i = -2.9999$$

Where: $\sum_{i=1}^{4} LE_i = -2.9999 \cong \nabla \cdot V = -3$, and the system is dissipative hyperchaotic.

Furthermore, system (2)'s maximal Lyapunov exponents (*MLE*) are higher than system (1) maximal Lyapunov exponents. This demonstrates that the recommended system outperforms system (1).

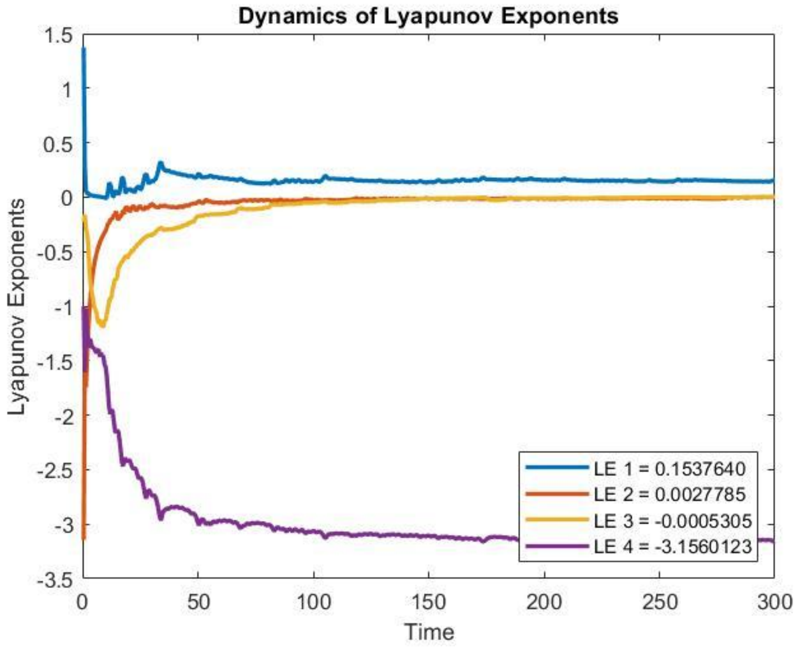

Figure 2. Lyapunov spectrum of a system (2) for parameters (4) and IC (5).

The novel hyperchaotic system (2) has the following Lyapunov dimension:

$$D_{LE} = \mathrm{J} + \frac{LE_1 + LE_2 + LE_3}{|LE_4|} = 3 + \frac{0.1538 + 0.0028 - 0.0005}{|-3.1560|} = 3.0495$$

Results

Table 1, which is shown in Fig. 3, shows that when IC (4) is implemented in System (2), with different typical parameters. it exhibits hyperchaotic, chaotic and Periodic behavior. Additionally, Table 2 presents a comparison between the new system (2) and several mathematical system models in terms of their equilibria, eigenvalues, and dynamic behavior.

Table 1. Lyapunov Exponents with different parameters with IC (4).

Figure	***Parameters*** $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$	$\boldsymbol{LE_1}$	$\boldsymbol{LE_2}$	$\boldsymbol{LE_3}$	$\boldsymbol{LE_4}$	***Sign of*** $\boldsymbol{LE_S}$	***Behavior***
Fig.3.a	$(1.5, 10, 0.1, 1)$	0.1654	0.0020	0.0002	-3.1676	$(+, +, 0, -)$	*Hyperchaotic*
Fig.3.b	$(1.5, 10, 0.06, 0.05)$	0.1345	-0.0006	-0.0189	-3.1151	$(+, 0, -, -)$	*Chaotic*
Fig.3.c	$(1.3, 10, 0.06, 1)$	0.2167	0.0001	-0.0050	-2.8118	$(+, 0, -, -)$	*Chaotic*
Fig.3.d	$(1.5, 10, 0.69, 1)$	-0.0003	-0.0330	-0.0535	-2.9132	$(0, -, -, -)$	*Periodic*

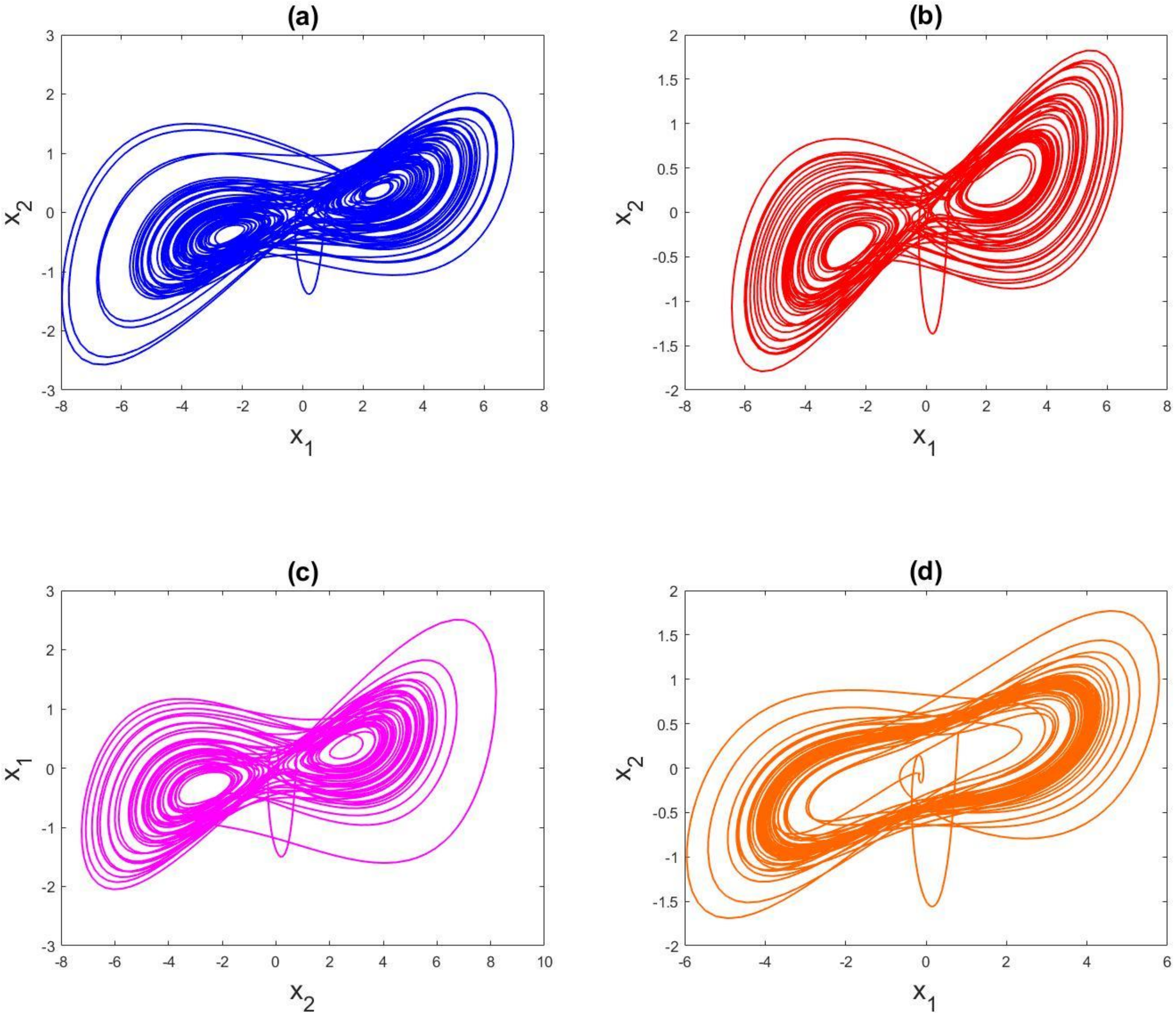

Figure 3. System of attractors under different parameters: (a) hyperchaotic, (b) chaotic, (c) chaotic, (d) periodic.

Table 2. Equilibria, eigenvalues, and behavior of several typical chaotic systems.

System	*Equation*	*Equilibria*	*Eigenvalue*	**No.+ve LE_s**
i. Vaidyanathan 2015[22]	$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = a(1-x_1^2)x_4 + bx_3 - x_1 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = c(1-x_3^2)x_2 + dx_1 - x_3 \end{cases}$	(0,0,0,0)	$\lambda_1 = -0.0667$ $\lambda_2 = -0.0526$ $\lambda_3 = -8.3809$ $\lambda_4 = +8.5002$	$(+,0,-,-)$
ii. Singh & Roy 2018[23]	$\begin{cases} \dot{x}_1 = x_1x_3 + x_2 \\ \dot{x}_2 = x_2x_3 - bx_1 + x_4 \\ \dot{x}_3 = 1 - x_1^2 - x_2^2 \\ \dot{x}_4 = -ax_2 \end{cases}$	$(\pm1,0,0,\pm b)$	$\lambda_{1,2} = \pm1.8478\,i$ $\lambda_{3,4} = \pm0.7654\,i$	$(+,0,0,-)$
iii. Dong & Wang 2022[24]	$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + kx_1x_3 + x_4 \\ \dot{x}_2 = -cx_2 - x_1x_3 \\ \dot{x}_3 = -b + x_1x_2 \\ \dot{x}_4 = mx_2 \end{cases}$	No- Equilibria Hidden	--------------	$(+,+,0,-)$
iv. Prakash et al. 2020[25]	$\begin{cases} \dot{x}_1 = a_1(x_2 - x_1) \\ \dot{x}_2 = -x_1x_3 + a_2x_2 - 5x_4 + 1 \\ \dot{x}_3 = x_1x_2 - a_5x_3 \\ \dot{x}_4 = a_7x_2 \end{cases}$	$(1/5, 0, 0, 0)$	$\lambda_1 = -30$ $\lambda_2 = 19.9732$ $\lambda_3 = 0.025$ $\lambda_4 = -2.9983$	$(+,+,0,-)$

v. **New system**	$\begin{cases} \dot{x} = -ax + yz \\ \dot{y} = -ay + xz - bx - cw \\ \dot{z} = 1 - xy \\ \dot{w} = x - dy \end{cases}$	$\begin{pmatrix} \sqrt{d}, \frac{1}{\sqrt{d}}, ad, \frac{-a + ad^2 - bd}{c\sqrt{d}}, \\ -\sqrt{d}, \frac{-1}{\sqrt{d}}, ad, \frac{a - ad^2 + bd}{c\sqrt{d}} \end{pmatrix}$	$\lambda_1 = -0.0269$ $\lambda_2 = -0.2520$ $\lambda_{3,4} = -1.6125 \mp 3.8823i$	$(+, +, 0, -)$

– Dissipative hyperchaotic system with two equilibria points.

– Dissipative hyperchaotic system with hyperbolic Saddle-Focus points.

– System has two positive Lyapunov exponents.

Circuit implementation

In this part, the electronic circuit design of a new system is built using Multisim 14.2 and includes operational resistors, amplifiers, capacitors, multipliers, and amplifiers, as seen in Fig. 4, to increase the validity of the experimental results. In this designed circuit, each amplifier and multiplier is an AD633 under voltage $\pm 15V$ and a TL082CD. Hardware design is crucial and important for chaotic systems because of its applications in chaos-based engineering [26–28].

Since $\tau = \tau_0\, t$ (where $\tau_0 = 1000$) and 10 represent the time scaling factor and amplitude scaling factor, respectively, the proposed system may be described within the parameters (3) as follows:

$$\begin{cases} \dot{x} = -1500\, x - 10000(-y)z \\ \dot{y} = -1500\, y - 10000(-x)z - 10000x - 60w \\ \dot{z} = -100(-1) - 10000(x)\, y \\ \dot{w} = -1000(-x) - 1000\, y \end{cases} \tag{10}$$

According to Kirchhoff's law, the associated circuit equations may be written as follows:

$$\begin{cases} \dot{x} = -\frac{1}{R_1C_1}x - \frac{1}{10R_2C_1}(-y)z \\ \dot{y} = -\frac{1}{R_3C_2}y - \frac{1}{10R_4C_2}(-x)z - \frac{1}{R_5C_2}x - \frac{1}{R_6C_2}w \\ \dot{z} = -\frac{1}{R_7C_3}(-V_0) - \frac{1}{10R_8C_3}x\,y \\ \dot{w} = -\frac{1}{R_9C_4}(-x) - \frac{1}{R_{10}C_4}y \end{cases} \tag{11}$$

Here, all capacitors have the value $C_i = 10\, nF, i = 1, 2, 3, 4$, when comparing (10) and (11), the following values are specified as the parameters for each circuit component:

$$R_1 = R_3 = 0.066M\Omega\,, \quad R_2 = R_4 = R_8 = 1k\Omega\,, \quad R_5 = 10k\Omega\,,$$

$$R_6 = 0.016\, M\Omega\,,\ R_7 = 1M\Omega\,,\ R_9 = R_{10} = 100\, k\Omega\,.$$

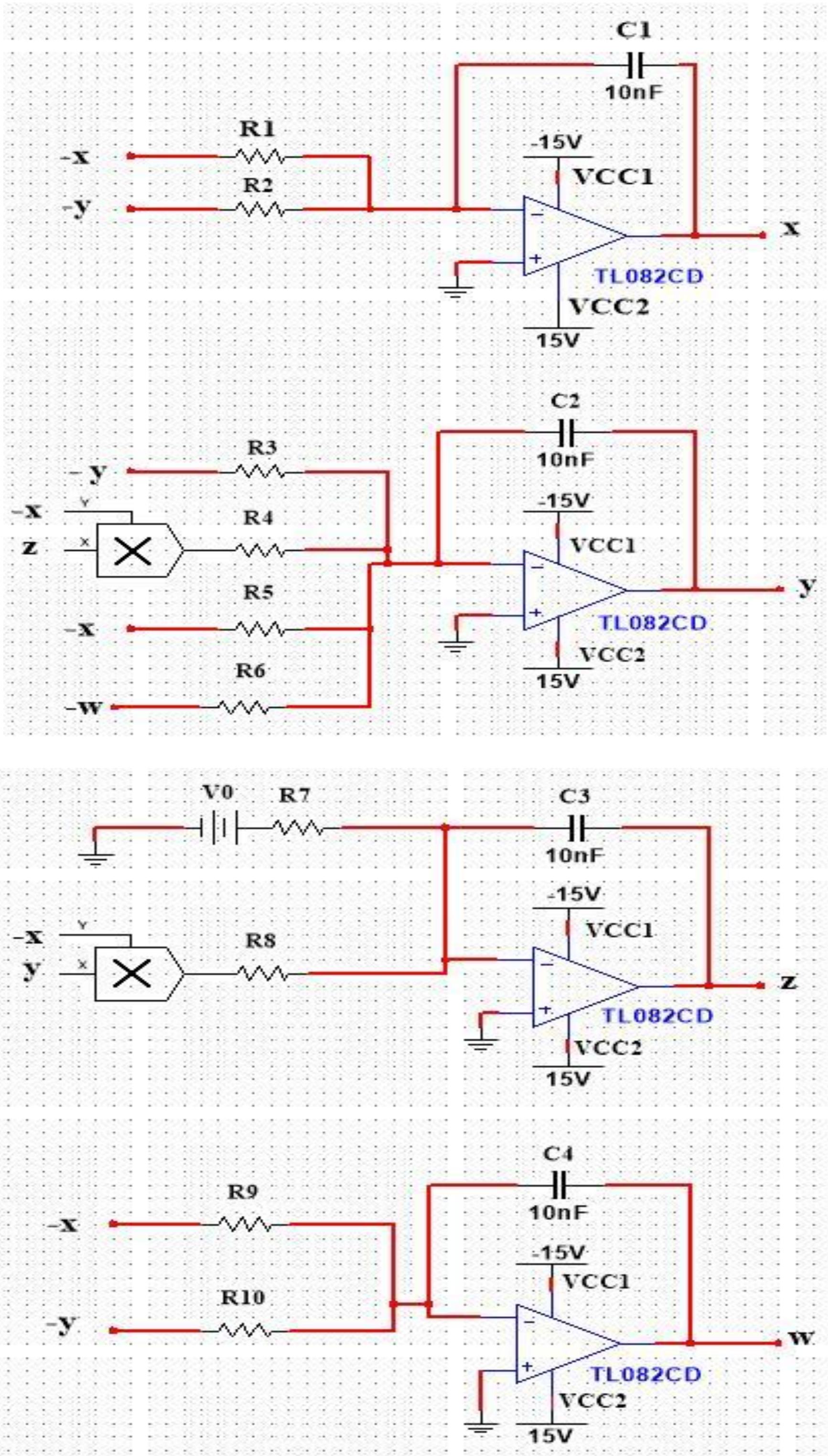

Figure 4. Circuit implementation using Multisim software for the suggested system.

The designed circuit was implemented and simulated in Multisim 14.2 using the specified component values. The simulation results show irregular oscillations that closely match the numerical behavior predicted by the theoretical model in Equation (10), confirming the successful implementation of the chaotic system [29]. Furthermore, the results of the analog circuit are compatible with digital hardware implementations such as FPGA platforms, supporting the system's feasibility in practical applications [30].

Conclusion

A new 4D system was obtained through feedback strategies. This system is considered simple; it consists of a few terms (ten terms only). In addition, the dynamic properties were analyzed numerically and analytically, obtaining Saddle-Focus equilibria points. In addition, it obtained several types of chaotic behavior, such as: hyperchaotic, chaotic, and periodic. Finally, an electronic circuit was implemented in the new system using Multisim 14.2 software. This system can be employed in various applications, such as secure communications, encryption, and neural modeling, due to its chaotic behavior and diverse dynamic characteristics.

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