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Original Article

Strongly Involution k-regular nil clean ring

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Abstract

This article is about A-rings R, for which every element is a sum of an involution k-regular element and a nilpotent element, which commute with each other. These rings are called strongly involution k-regular nil-clean rings or SIKRNC rings. Some characteristics and basic properties of SiKrnc rings are presented in this work.

Keywords: Clean rings, Nil clean rings, Strongly involution m-regular nil clean rings

Introduction

In this article, we assume that all rings are associative and have an identity element. In [1], [2], [3],[4], ring $\mathcal{R}$ is considered to be clean. Every element is defined in terms of sum of unit $u \in \mathcal{R}$ and idempotent $e \in \mathcal{R}$. Moreover, Diesl and Alexander studied the nil clean ring [3], which showed that R is nil clean if all elements are idempotent $e \in \mathcal{R}$ and nilpotent $c \in \mathcal{R}$. An elementd□R is called regular element, if there is an element l□R and positive integer k such that dk=dkldk. if every elements in R k-regular, then R is called k-regular[5], when $k = 1$, that is $d = d\ \ell\ d$, $\mathcal{R}$ is called Von Neumann regular ring (simply regular) [5], [6]. An element v∈R is considered to be involution if it is a square roots of 1 [7], [8].

Any ring $\mathcal{R}$ may be represented by N($\mathcal{R}$), $U(\mathcal{R})$, J($\mathcal{R}$), Idem($\mathcal{R}$), k - reg($\mathcal{R}$), Inv($\mathcal{R}$), and Ik- reg($\mathcal{R}$) to represent the set of nilpotent elements, the group of unite elements, the Jacobson radical of $\mathcal{R}$, the set of idempotent element in $\mathcal{R}$, the set of k - regular elements of $\mathcal{R}$, all involutional elements and all k-regular elements respectively. Finally, $\mathcal{R}$ is referred reduced if N($\mathcal{R}$)= {0}, [9].

SImrnc -Rings

A definition of k – regular nil clean rings is present and SImrnc – rings with some characteristics and fundamental properties.

Definition *1* [10]

An element $x \in \mathcal{R}$ is k-regular nil clean or (k-rnc) when $x = d^k + c$ where $d^k \in k - reg(\mathcal{R})$ for a fixed positive integer $k \geq 1$ and $c \in$ N($\mathcal{R}$). A ring $\mathcal{R}$ is k-regular nil clean ring, or (k-rnc ring) if each element in $\mathcal{R}$ is k-rnc.

Example [10]

1. $\mathcal{Z}_6$ is k- rnc ring but not nil clean.
2. $\mathcal{Z}_8$ is k- rnc ring when $k = 3$ but, it is not regular.

Definition 2

Any element of ring $\mathcal{R}$ can be expressed as a sum of a ring of evolution k-regular, nilpotent elements, then it is called to be an Involution k-regular, nil clean ring (simply I$k$rnc – ring).

The following is an example for I$k$rnc – ring

① Let $\mathcal{R} = \{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathcal{Z}_2\}$

$\mathcal{A}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathcal{A}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{A}_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\mathcal{A}_5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{A}_6 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{A}_7 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$x = d^k + c = d^k \ \mathcal{V} \ \ d^k + b, \ \ inv(\mathcal{R}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\mathcal{A}_4 = \mathcal{A}_1 + \mathcal{A}_2$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\mathcal{A}_2 = \mathcal{A}_2 + \mathcal{A}_2$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\mathcal{A}_6 = \mathcal{A}_3 + \mathcal{A}_2$

$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$\mathcal{A}_1 = \mathcal{A}_4 + \mathcal{A}_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\mathcal{A}_5 = \mathcal{A}_5 + \mathcal{A}_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathcal{A}_3 = \mathcal{A}_6 + \mathcal{A}_2$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathcal{A}_7 = \mathcal{A}_7 + \mathcal{A}_0$

$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$

② $(\mathcal{Z}_p, +, \cdot)$, $(\mathcal{Z}_2^h, +, \cdot)$, $(\mathcal{Z}_3^h, +, \cdot)$

Now, we give the definition of Simrnc – ring.

Definition 3

A ring $\mathcal{R}$ is defined as strongly Involution k-regular nil clean (simply Simrnc – ring), when any element x in $\mathcal{R}$, exists ivolution k -regular element $d^k \in \mathcal{R}$ and a nilpotent member c in $\mathcal{R}$ such that $x = d^k + c$, with $d^k c = c d^k$.

Example: $(\mathcal{Z}_2^h, +, \cdot)$ and $(\mathcal{Z}_3^h, +, \cdot)$, where h is positive integer, are I$k$rnc rings.

Proposition 1

Every homorphic image of Simrnc ring is Simrnc too.

Proof: -

$H: \mathcal{R} \to \mathcal{R}_1$ is a homomrphism from Simrnc ring $\mathcal{R}$ onto the ring $\mathcal{R}_1$. For any $\varphi \in \mathcal{R}_1$, there exist. $x \in \mathcal{R}$ such that $\varphi = g(x)$ since $x = d^k + c$, where $d^k \in Ikreg\,(\mathcal{R})$ and $c \in \mathrm{N}(\mathcal{R})$, thus $\varphi = \mathrm{H}(x) = \mathrm{H}(d^k + c) = \mathrm{H}(d^k) + \mathrm{H}(c)$.Let $y = \mathrm{H}(d)$, then $y^k = ((d))^k = g(d^k \mathcal{V} d^k) = \mathrm{H}(d^k)\,\mathrm{H}(\mathcal{V})\,\mathrm{H}(d^k) = (\mathrm{H}(d))^k \mathrm{H}(\mathcal{V})(\mathrm{H}(d))^k = y^k\,\mathrm{H}(\mathcal{V})\,y^k$. Now$\mathrm{H}(\mathcal{V}).\mathrm{H}(\mathcal{V}) = \mathrm{H}(\mathcal{V}.\mathcal{V}) = \mathrm{H}(1) = 1'$, and $\mathrm{H}(\mathcal{V})$ is involution element in $\mathcal{R}_1$, $\mathrm{H}(d^k)$ is Invo. k-regular in $\mathcal{R}_1$. Since $(\mathrm{H}(c))^n = \mathrm{H}(c^n) = \mathrm{H}(0) = 0$, imples that $\mathrm{H}(c) \in \mathrm{N}(\mathcal{R}_1)$. Because $\mathrm{H}(d^k).\mathrm{H}(c) = \mathrm{H}(d^k.c) = \mathrm{H}(c.d^k) = \mathrm{H}(c).\mathrm{H}(d^k)\varphi$ is Simrnc element in $\mathcal{R}_1$. ■

Proposition 2

Let R be a reduced SIkrnc - ring. Then 6=0.

Proof: -

Let 1+1=2 $\in \mathcal{R}$, then 2 can be written as $2 = d^k + c$, where $d^k \in Ikreg\,(\mathcal{R})$, $c \in \mathrm{N}(\mathcal{R})$, then $2 = d^k\,\mathcal{V}\,d^k$, where $\mathcal{V} \in Invo(\mathcal{R})$ and hence $2 = e\mathcal{V}, e \in Id(\mathcal{R})$, implies that $2\mathcal{V}^{-1} = e$. Thus $2\mathcal{V} = e$ and squaring, we get $(2\mathcal{V})^2 = e^2$ *implies that* $2^2\,\mathcal{V}^2 = e^2$ *tehen* $4(1) = e$. Again squaring $4^2 = e^2$ that is $4 = e$, hence $16 = 4$ that is $12 = 0$. Thus $6^2 = 36 = 3.12 = 3.0 = 0$. Hence $6^2 = 0$ and so $6 \in \mathrm{N}(\mathcal{R}) = \{0\}$. ■

Proposition 3

Let $\mathcal{R}$ is SImrnc ring with all central idempotent elements of $\mathcal{R}$, then $e\mathcal{R}e$ is SImrnc – ring.

Proof: -

Let $x \in e\mathcal{R}e \subset \mathcal{R}$, then $x = d^k + c$, where $d^k \in Ikreg(\mathcal{R})$ and $c \in \mathrm{N}(\mathcal{R})$. Hence $d^k = d^k.\mathcal{V}.d^k$, $\mathcal{V} \in Inv(\mathcal{R})$ implies that, $e\,x\,e = ed^ke + ece$, $e \in Id(\mathcal{R})$. Then $e\,d^k\,e = e.d^ke\,\mathcal{V}.e\,d^k$. Thus $e\,d^ke \in kr(\,e\mathcal{R}e)$. Now, to explain that$U(e\mathcal{R}e) = Inv(e\mathcal{R}e)$, let $u \in U(e\mathcal{R}e)$, we get that $u + 1 - e \in U(\mathcal{R})$ with inverse $u + 1 - e$, where $u\,u' = u'u = e$. Since, $u(1-e) = (1-e)u = 0$ and so $1 = (u + 1 - e)^2 = u^2 + 2u(1-e) + (1-e)^2 = u^2 + (1-e)$ indicates that $u^2 = e$. Hence $u(e\mathcal{R}e) = Inv(e\mathcal{R}e)$, also $(ece)^n = e^n c^n e^n = 0$ for some $n \in \mathcal{Z}^+$, that is $ece \in \mathrm{N}(e\mathcal{R}e)$. Therefore $x = (ed^k)\mathcal{V}(d^ke) + ebe$ is SImrnc element and, therefore ebe is SImrnc - ring.■

Proposition *4*

if $\mathcal{R}$ is a reduced ring. Then $\mathcal{R}$ is SImrnc ring if and only if $\mathcal{R} \cong \mathcal{Z}_2 \times \mathcal{Z}_3$.

Proof: -

If $\mathcal{R}$ is SImrnc and $x \in \mathcal{R}$,x can be written as $x = d^k + c$, where $d^k = d^k.\mathcal{V}.d^k$ for fixed positive integer k and $\mathcal{V}^2 = 1$, $c \in \mathrm{N}(\mathcal{R}) = O.Sox = d^k$, implies that $x^2 = d^{2k} = d^k\mathcal{V}$ and $x^3 = d^{3k} = (\mathcal{V}).d^k = d^k$. Therefore, from a classical theorem Jacobson, we get $\mathcal{R}$, is commutative in [11] [11. **Proposition 2.11**]. $J(\mathcal{R}) \subset \mathrm{N}(\mathcal{R}) = \{0\}$, that is $J(\mathcal{R}) = \cap_{M \in Max(\mathcal{R})} M = \{0\}$. Now, the function $f: \mathcal{R} \longrightarrow \pi_{M \in Max(\mathcal{R})}\,(\mathcal{R}/M)$ are homomorphism with Kernel $\cap_{M \in Max(\mathcal{R})} M = J(\mathcal{R})$. That is, we have an injection $\mathcal{R}/J(\mathcal{R})$ implies

$\pi_{M \in Max(\mathcal{R})}\,(\mathcal{R}/M)$, since $\mathcal{R} \cong \mathcal{R}/J(\mathcal{R})$, that is $\mathcal{R} \cong \pi_{M \in Max(\mathcal{R})}\,(\mathcal{R}/M)$. Because $\mathcal{R}/M$ is a field, it implies that M is a maximal ideal, $\mathcal{R}$ is SImrnc ring, then by **Proposition 1** we get $\mathcal{R}/M$ is SImrnc ring and a field must be isomorphic to $\mathcal{Z}_2 \times \mathcal{Z}_3$.

Conversely, since every element x in $\mathcal{Z}_2 \times \mathcal{Z}_3$ can be expressed as $d^{2k}\ \mathcal{V} + 0$ for $d^{2k} \in k.reg(\mathcal{R})$ and $\mathcal{V} \in Inv.(\mathcal{R})$. ■

Lemma 1

If e is an idempotent element in SImrncr reduced ring, then $e\mathcal{R} \cap d^k\mathcal{R} \subseteq ed^k\mathcal{R}$. For every $d^k \in Imr(\mathcal{R})$ and if $ed^k = 0$, then $e\mathcal{R} \cap d^k\mathcal{R} = 0$.

Proof: -

Let $x = d^k + c$, where $d^k \in Imr(\mathcal{R})$ and $c \in \mathrm{N}(\mathcal{R}) = \{0\}$. Hence $x = d^k.\mathcal{V}.d^k$ since $d^k\mathcal{V} = e$, then $d^k\mathcal{V}\mathcal{R} \cap x\,\mathcal{R} \subseteq (d^k\mathcal{V})x\,\mathcal{R}$ for every $x \in \mathcal{R}$. If $d^k\mathcal{V}\,x = 0$, then $d^k\mathcal{V}\mathcal{R} \cap d^k\,\mathcal{R} = 0$ and therefore $e\mathcal{R} \cap d^k\mathcal{R} = 0$. ■

Proposition *5*

A reduced ring $\mathcal{R}$ is SImrnc if and only if for any element x in $\mathcal{R}$, exists $d^k \in Imr(\mathcal{R})$ and $\mathcal{V} \in Inv(\mathcal{R})$ such that $x\,\mathcal{R}\ \cap (x - \mathcal{V})\mathcal{R} = 0$.

Proof: -

Let $\mathcal{R}$ is SImrnc ring, $x \in \mathcal{R}$, then x can be written as $x = d^k + c$, for $d^k \in Ikr(\mathcal{R})$ and $c \in \mathrm{N}(\mathcal{R}) = \{0\}$. Hence $x = d^k\ \mathcal{V}\ d^k$, $\mathcal{V} \in In(\mathcal{R})$ implies that $x = d^k\ \mathcal{V}^{-1}\ d^k$, then $x = d^k = d^k\ \mathcal{V}^{-1}\ d^k$. Thus $d^k\ \mathcal{V}^{-1}(\ d^k - \mathcal{V}) = 0$. Since $d^k\mathcal{V} \in Id(\mathcal{R})$, $d^k\ \mathcal{V}^{-1}\ \mathcal{R} \cap (\ d^k - \mathcal{V})\mathcal{R} = 0$.

Conversely $x\,\mathcal{R}\ \cap (x - \mathcal{V})\mathcal{R} = 0$, so $d^k\ \mathcal{V}^{-1}\ \mathcal{R}\ \cap (\ d^k - \mathcal{V})\mathcal{R} = 0$ where $d^k\ \mathcal{V} \in Id(\mathcal{R})$, $(\mathcal{V}^{-1} = \mathcal{V})$, since $d^k\ \mathcal{V}^{-1}(\ d^k - \mathcal{V}) = 0$ indicates $x = d^k\ \mathcal{V}^{-1}\ d^k = d^k$ where $\mathcal{V} \in Inv(\mathcal{R})$. Hence $x = d^k\ \mathcal{V}\ d^k$ when $d^k \in Imr(\mathcal{R})$ and $c \in \mathrm{N}(\mathcal{R}) = \{0\}$. Therefore x is SImrnc. ■

SImrnc -rings Z2h

We will establish some characteristics of $(\mathcal{Z}_2^h, +, \cdot)$ rings.

Proposition 6

For every element x in $(\mathcal{Z}_2^h, +, \cdot)$, $x - x^2$ is nilpotent element, if $2 \in \mathrm{N}(\mathcal{R})$.

Proof: -

Since $(\mathcal{Z}_2^h, +, \cdot)$ SIkrnc - ring , then x can be written as $x = d^k + c$, where $d^k = d^k.\mathcal{V}.d^k$ for fixed positive integer $k \geq 1$ and $\mathcal{V}^2 = 1$, $c \in \mathrm{N}(\mathcal{R})$.

Now, $x^2 = (d^k + c)^2 = d^k\mathcal{V} + 2d^kc + c^2$

$= d^k\mathcal{V} + 2d^kc.$

$x - x^2 = d^k + c - d^k\mathcal{V} - 2d^kc$

$= d^k(1 - \mathcal{V}) + (1 - 2d^k)c.$

Since $d^k(1 - \mathcal{V})$ and $(1 - 2d^k)c$ are nil potent elements. ■

Proposition 7

Let x be a non - zero divisor in $(\mathcal{Z}_2^h, +, \cdot)$. Then x is a unit element if $2 \in U(\mathcal{R})$.

Proof: -

When $x \in \mathcal{R}$ is a non - zero divisor is in the ring $(\mathcal{Z}_2^h, +, \cdot)$. From **Proposition 6**, we have $x^2 - x \in \mathrm{N}(\mathcal{R})$ follows that $x^2 - x \in \mathrm{N}(\mathcal{R})$, then there $r \in Z^+$such that $(x^2 - x)^r = 0$ so $x^r(x - 1)^r = 0$. Since x^r is not a zero divisor , so $(x - 1)^r = 0$ that is $(x - 1) \in \mathrm{N}(\mathcal{R})$. So $n$implies that $x = n + 1$, put $n + 1 = u$, thus $x = u$. Therefore, every non – zero divisor is a unit. ■

Proposition 8

In the ring $(\mathcal{Z}_2^h, +, \cdot)$, $J(\mathcal{R})$ is nil ideal, if $2 \in \mathrm{N}(\mathcal{R})$.

Proof: -

Let $x \in J(\mathcal{R})$, then $x = d^k + c$, where $d^k = d^k\, \mathcal{V}\, d^k$ for fixed positive integer k and $\mathcal{V}^2 = 1$, $c \in \mathrm{N}(\mathcal{R})$, $d^k\, \mathcal{V} = \mathcal{V}\, d^k$, $d^k\, c = c\, d^k$. So $(1 - x)$ is unit and from **Proposition 6** we have $(x - x^2) \in \mathrm{N}(\mathcal{R})$, assume that $\mathcal{W} = (x - x^2) \in \mathrm{N}(\mathcal{R})$, which implies that $\mathcal{W} = x(1 - x)$ and hence $x = \mathcal{W}(1 - x)^{-1} \in \mathrm{N}(\mathcal{R})$.Therefore, $J(\mathcal{R})$ is nilpotent ideal.■

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Competing Interests

The authors affirm that there are no conflicts of interest pertaining to the publication of this paper.

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