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Original Article

On Ph-Continuous Functions and T_i^{ph}- Spaces

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ABSTRACT

In this paper, we introduce a set of new and significant functional concepts in topology, including ph-continuous functions, ph-open functions (ph-openness), and ph-irresolute functions. The fundamental characteristic properties of these functions are investigated, in addition to reviewing and defining the conditions for the concept of ph-homeomorphism. Furthermore, we establish and define a new set of ph-separation axioms that enhance the precision of classifying topological spaces. Subsequently, we analyse the interrelationships among these axioms and their points of distinction from the standard axioms.

Keywords: *ph-continuity, ph-openness, ph-irresolute, ph-homeomorphism, ph-separation axioms*

Introduction

In a study [10], introduced and studied topological properties of pre-continuous function; and h-continuous function was introduced by [1]; Askander [3] Biswas [4] Mashhour, Hasanein, and El-Deeb [11] Crossley [5] Maheshwari [8] Based on the definitions of open sets they provided and examined different classes of continuity and other properties of functions in topological spaces. Munshi [12] proposed separation axioms. Many topologists studied the separation axioms [6-8,13 and 14].

This study is an extension of a previous work (as cited in [2]) centered on ph-open sets, which constitute a generalization of both standard open sets and h-open sets. Furthermore, they share certain properties with pre-open sets in topological spaces. In this research, we present an extensive investigation of functions defined on these sets, with a particular focus on the concept of ph-continuity for these functions. We identify the sets that preserve their properties under this type of continuity, in addition to exploring the notion of ph-homeomorphism and the necessary conditions for the transfer of other topological properties. Finally, the ph-separation axioms defined based on ph-open sets are studied, specifically addressing the analysis of T_0^{ph}, T_1^{ph} and T_2^{ph} spaces.

Materials & Methods

In this paper, we use the Mathematical logic, alongside other theories, was used as the foundation for the proofs of problems and theorems

2. Ph-Continuous Functions

In this section, we introduced a new further type of ph-continuous (ph-cont), ph-open (ph-o) and ph-homeomorphism functions and studied some properties of these functions.

Definition 2.1 [2]: For the topological space (T.S.) (X, τ), the set $S \subseteq X$ called pre-h-open (ph-os) if it is contained in the interior of its h-closure, which is expressed as $S \subseteq int(cl_h(S))$. The ph-o sets' complement is named pre-h-closed (ph-c). We will put τ^{ph} to indicate the collection of all ph-o sets defined in (X, τ).

Definition 2.2: A function $f: (X, \tau_1) \to (Y, \tau_2)$ is ph-open (ph-o), if $f(A)$ is (ph-os) in Y, for any open set (os) A in X.

Example 2.3: For $R = \{1,3,2\} = S$ and $\tau = \{\{2,3\}, \{2\}, \{1,2\}, R, \emptyset\}$,

$\sigma = \{S, \{2\}, \{1,3\}, \emptyset\}$, $\sigma^{ph} = \{\{2\}, \{2,3\}, \{1,2\}, \{3\}, \{2\}, \{1,3\}, \emptyset, S\}$.

and $f: (R, \tau) \to (S, \sigma)$ is the identity function. Clearly f is (ph-o).

Theorem 2.4: If $f: (X, \tau_1) \to (Y, \tau_2)$ is open function, then f is (ph-o) function.

Proof: Let S be (os) in X. Since the function f is open, then $f(S)$ is (os) in Y. By (Theorem 2.5, [2]), $f(S)$ is (ph-os) in Y. Hence, f (ph-o). ■

Note: *The opposite of the above theorem is incorrect.*

Example 2.5: From Example (2.3) f is (ph-o) but not (os).

Proposition 2.6: Every (ph-o) function is p-open (p-o).

Proof: It is clear.

Note: Clearly, the opposite of the previous proposition is incorrect.

Example 2.7: Let $X = \{2,1,3\} = Y$, $\tau = \{X, \{1\}, \{1,3\}, \emptyset\}$, $\sigma = \{Y, \{1,2\}, \emptyset\}$, $\sigma^p = \{\{2\}, \{2,3\}, \{1\}, \{2,3\}, \emptyset, Y\}$, $\sigma^{ph} = \{Y, \{1,2\}, \emptyset\}$ and $f: (X, \tau) \to (Y, \sigma)$ is an identity function. Hence f is not (ph-o) function.

Proposition 2.8: If $f: (X, \tau_1) \to (Y, \tau_2)$ is open function and $h: (Y, \tau_2) \to (Z, \tau_3)$ be is (ph-o), then $h \circ f: (X, \tau_1) \to (Z, \tau_3)$ is (ph-o).

Proof: Consider any (os) A in X. Because f is an open function, $f(A)$ is also (os) in Y. By (Theorem 2.5, [2]), $f(A)$ is (ph-os) in Y. Since h is (ph-o), then $(h \circ f)(A) = h(f(A))$ is (ph-os) in Z. Hence $h \circ f$ is (ph-o). ■

Next, we provide the next definition:

Definition 2.9: The function $f: (R, \tau_1) \to (S, \tau_2)$ named ph-continuous (ph-cont). If $f^{-1}(A)$ is (ph-os) in R for an (os) A in S.

Example 2.10: Let $R = \{b, a, c\}$ and $S = \{3,1,2\}$ $\tau = \{\{b\}, \emptyset, R\}$,

$\tau^{ph} = \{\{a, b\}, \emptyset, R, \{b\}\}$ and $\sigma = \{\{1\}, \{1,3\}, \emptyset, S\}$. Let $f: (R, \tau) \to (S, \sigma)$ is the ideality function. Then f is (ph-cont).

Proposition 2.11: Let the function $f: (X, \tau_1) \to (Y, \tau_2)$ be (cont), then f is (ph-cont).

Proof: Consider an open subset A in Y. Since f is (cont), then $f^{-1}(A)$ is (os) in X. According to Theorem 2.5, [2], the $f^{-1}(A)$ is a (ph-os) in X. Therefore, f is (ph-cont). ■

Note: The converse of the previous proposition is incorrect.

Example 2.12: Let $R = \{c, a, b\}$, $\tau = \{\{b\}, R, \emptyset\}$, $\tau^{ph} = \{\{b, c\}, \{b\}, \emptyset, \{a, b\}, R\}$, $S = \{3,1,2\}$ $\sigma = \{S, \emptyset, \{1,3\}\}$. A function $T: (R, \tau) \to (S, \sigma)$ is defined by $T(a) = 2$, $T(b) = 1$, $T(c) = 3$. Clearly, T is (ph-cont) but not (cont) function.

Proposition 2.13: Let the function $f: (X, \tau_1) \to (Y, \tau_2)$ be (ph-cont), then f is p-(cont) function.

Proof: Since f is (ph-cont) function, then by Definition (2.9). Let G be an (os) in Y, then the inverse image of G is (ph-os) in X, by (Theorem (2.10), [2]) the inverse image of G is (p-os) in X. Hence f is p-(cont). ■

Note: The opposite of the previous proposition is incorrect.

Example 2.14: Let $R = \{b, c, a\}$, $\tau = \{\{b, a\}, R, \emptyset\}$,

$\tau^p = \{\{b, c\}, R, \{b, a\}, \emptyset, \{c, a\}, \{a\}, \{b\}\}$ $\tau^{ph} = \{R, \emptyset, \{b, a\}\}$, $S = \{3,1,2\}$, $\sigma = \{\{2,1\}, \emptyset, S\}$, and $f: (R, \tau) \to (S, \sigma)$ be defined as $f(a) = 3$, $f(c) = 2$, $f(b) = 1$. Clearly, f is p-(cont) but not (ph-cont) function.

Theorem 2.15: Let $f: (X, \tau_1) \to (Y, \tau_2)$ is (ph-cont) and $h: (Y, \tau_2) \to (Z, \tau_3)$ is (cont), then $h \circ f$ is (ph-cont).

Proof: Consider any open subset A in Z. Since we have h is (cont) function then $h^{-1}(A)$ is (os) in Y. Since f is (ph-cont), then $f^{-1}(h^{-1}(A)) = (h \circ f)^{-1}(A)$ is (ph-os) in X. Therefore $h \circ f$ is (ph-cont) function. ■

Definition 2.16: The function $f: (R, \tau_1) \rightarrow (S, \tau_2)$ named ph-irresolute. If $f^{-1}(A)$ is (ph-os) in R for every (ph-os) A in S.

Example 2.17: Let $R = \{c, b, a\}$ and $S = \{3,1,2\}$ $\tau = \{\{b\}, \emptyset, R\}$,

$\tau^{ph} = \{\{a, b\}, \emptyset, R, \{b\}\}$ and $\sigma = \{\{1\}, \{1,3\}, \emptyset, S\}$ and $\sigma^{ph} = \{\{1,3\}, \emptyset, \{1\}, S\}$. Let $f: (R, \tau) \rightarrow (S, \sigma)$ is the ideality function. Then f is ph-irresolute.

Theorem 2.18:

1) Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be (cont) function, then f is ph-irresolute.
2) Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be ph-irresolute function, then f is (ph-cont).

Proof (1): Consider any (ph-os) A in Y and since we have f is a (cont). Then $f^{-1}(A)$ is (os) in X and by Theorem 2.3 the $f^{-1}(A)$ is (ph-os) in X. Therefore, f is ph-irresolute.

(2) Consider any open subset A in Y and f is ph-irresolute. We have A is (ph-os) by (Theorem 2.5, [2]), since f is ph-irresolute, then $f^{-1}(A)$ is (ph-os) in X. So, f is (ph-cont). ■

Theorem 2.19: Let $f: (X, \tau) \rightarrow (Y, \vartheta)$ and $h: (Y, \vartheta) \rightarrow (Z, \eta)$ are ph-irresolute, then $h \circ f: (X, \tau) \rightarrow (Z, \eta)$ is ph-irresolute.

Proof.:Consider any (ph-os) A in Z. Since h is ph-irresolute, then $h^{-1}(A)$ is (ph-os) in Y. Since f is ph-irresolute, then $f^{-1}(h^{-1}(A)) = (h \circ f)^{-1}(A)$ is (ph-os) in X. Hence, $h \circ f$ is ph-irresolute. ■

Definition 2.20: Let $f: (X, \tau_1) \rightarrow (Y, \tau_2)$ be a bijective function then it is ph-homemorphism if f is ph- (cont) and (ph-o) function.

Example2.21: Let $X = \{c, a, b\}$, $\tau = \{X, \{b\}, \emptyset, \{b, a\}\}$, $\tau^{ph} = \{X, \{b\}, \emptyset, \{b, a\}\}$, $Y = \{3,1,2\}$, $\sigma = \{\{2\}, \{1,3\}, \emptyset, Y\}$, $\sigma^{ph} = \{\emptyset, Y, \{3\}, \{1\}, \{2\}, \{2,1\}, \{3,1\}, \{3,2\}\}$ and $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined as $f(b) = 3$, $f(a) = 1$, $f(c) = 2$. Hence f is ph-homeomorphism.

Theorem 2.22: Every homeomorphism function is ph-homeomorphism.

Proof: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be homeomorphism function. Then f is (cont) and by Proposition (2.10) f is the (ph-cont) and by Theorem (2.3), f is (ph-o) function. Since f is bijective. Then f is ph- homeomorphism function. ■

The opposite of the previous theorem is incorrect.

Example2.23: Let $X = Y = \{3,1,2\}$, $\tau = \{X, \{1,3\}, \emptyset, \{1\}\} = \tau^{ph}$, $\sigma = \{\{1\}, \emptyset, Y\}$, $\sigma^{p} = \{\{1,2\}, \emptyset, \{1\}, \{1,3\}, Y\}$ and $f: (X, \tau) \rightarrow (Y, \sigma)$ be the ideality function. Hence f is not homeomorphism.

Proposition 2.24: Every ph- homeomorphism function is p-homeomorphism.

Proof: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be ph- homeomorphism function. Then f is (ph-cont) function, (ph-o) and bijective function by Proposition (2.12), f is p-(cont) and by Proposition (2.5), f is p-open and f is bijective. Hence f is p- homeomorphism function. ■

The opposite of the above proposition is incorrect.

Example 2.25: Let $X = \{c, a, b\}$, $\tau = \{\emptyset, \{a, b\}, X\}$,

$\tau^{p} = \{\{a\}, \{b, c\}, \emptyset, X, \{b, a\}, \{b\}, \{c, a\}\}$ $\tau^{ph} = \{\emptyset, \{a, b\}, X\}$, $Y = \{3,1,2\}$, $\sigma = \{\{2,1\}, \emptyset, Y\}$, and $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined as $f(b) = 1$, $f(a) = 3$, $f(c) = 2$. Clearly, f is p-homeomorphism but not ph-homeomorphism function.

3- ph-Separation Axioms

In this section, we prove some results on T_i^{ph}-space, $i = 0,1,2$, we recall the following definition:

Definition 3.1: A T.S. (X, τ) is referred to as T_0^{ph}-space, if for every $a, b \in X$, $a \neq b$, there exists (ph-os) containing one but not containing the other.

Definition 3.2: A T.S. (X, τ) is referred to as T_1^{ph}-space if for every $a, b \in X$, $a \neq b$ there exists a pair of (ph-o) sets, one containing a but not b, and the other containing b but not a.

Definition 3.3: A T.S. (X, τ) is referred to as T_2^{ph}-space if for every $a, b \in X$, $a \neq b$ there exists a pair of disjoint (ph-o) sets, one containing a, and the other containing b.

Theorem 3.4: Every T_0-space is T_0^{ph}-space.

Proof: Let (X,τ) is T_0-space and $x,y \in X$, $x \neq y$. Then there exists (os) $A \subseteq X$. Such that $x \in A, y \notin A$. By (Theorem 2.5, [2]) every (os) is (ph-os). Hence X is T_0^{ph}-space. ■

Theorem 3.5: Every T_1-space is T_1^{ph}-space.

Proof: Let (X,τ) is T_1-space and $x,y \in X$, $x \neq y$. Then there exists two open sets A, B in X such that $x \in A, y \notin A$ and $x \notin B, y \in B$. By (Theorem 2.5, [2]) every (os) is (ph-os). Therefore, X is T_1^{ph}-space. ■

Theorem 3.6: Every T_2-space is T_2^{ph}-space.

Proof: Let (X,τ) is T_2-space and $x,y \in X$, $x \neq y$. Then there exists two disjoint open sets A, B in X such that $x \in A$ and $y \in B$. By (Theorem 2.5, [2]) every (os) is (ph-os). Therefore, X is T_1^{ph}-space. ■

Theorem 3.7:

1) Every T_1^{ph}-space is T_0^{ph}-space.
2) Every T_2^{ph}-space is T_1^{ph}-space.

Proof: (1) and (2) are Clear.

Theorem 3.8: A T.S. (X,τ) is T_0^{ph}-space if and only if $\{a\} \neq \{b\}$ for every pair of different points $a, b \in X$.

Proof: Let a, b be any two different points of T_0^{ph}-space X. We show that $\{a\}^* \neq \{b\}^*$. By hypothesis, assume that A is (ph-os) such that $a \in A$ and $b \notin A$. Hence $b \in X - A$ and $X - A$ is ph-closed set. Therefore, $\{b\}^* \subset X - A$. Hence $b \in \{b\}^*$, $a \notin X - A$. Hence $\{a\}^* \neq \{b\}^*$.

Conversely: assume that for all $a, b \in X$ with $a \neq b$, $\{a\}^* \neq \{b\}^*$. Now, let $z \in X$ such that $z \in \{a\}^*$ but $z \notin \{b\}^*$. If $a \in \{b\}^*$ then $\{a\} \subset \{b\}^*$ which implies that $\{a\}^* \subset \{b\}^*$. Thus $a \in \{a\}^*$ and $z \notin \{b\}^*$. This is contradiction. Therefore, $a \in \{b\}^*$. Hence $X - \{b\}^*$ is (ph-os) containing a but not b. So X is T_0^{ph}-space. ■

Theorem 3.9: Every T_1^{ph}-space is T_1^{p}-space.

Proof: Let a space X is T_1^{ph}-space and $x,y \in X$, $x \neq y$. Then there exists (ph-o) sets A, B such that $y \notin A, x \in A$ and $y \in B, x \notin B$. By (Theorem (2.10), [2]) every (ph-os) is p-open. Hence X is T_1^{p}-space. ■

We have the following relations on T_i^{ph}-space, $i = 0,1,2$ see figure 1.

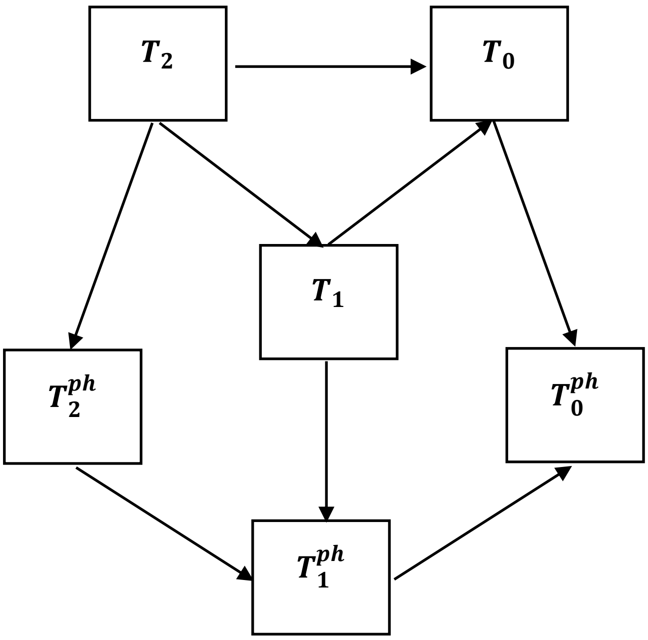

Figure 1. Relation with separation axiom

Results & Discussion

In this research, it was proven that every open function is necessarily a ph-open function, with a counterexample provided to show that the converse is not true. Similarly, it was established that every ph-open function is a p-open function, while the converse also remains false, which was further supported by a counterexample. Regarding the concept of continuity, the results indicated that a continuous function is ph-o continuous, but the converse does not hold. Finally, within the context of Separation Axioms, the study concluded the following inclusion relations: Every $\boldsymbol{T_0}$-space is $\boldsymbol{T_0^{ph}}$-space, every $\boldsymbol{T_1}$-space is $\boldsymbol{T_1^{ph}}$, eery $\boldsymbol{T_2}$-space is $\boldsymbol{T_2^{ph}}$, every $\boldsymbol{T_1^{ph}}$-space is $\boldsymbol{T_0^{ph}}$-space, every $\boldsymbol{T_2^{ph}}$-space is $\boldsymbol{T_1^{ph}}$-space, noting that all converse relations are not true.

Conclusion

This study presents an extension of the concept of h-open sets and their associated functions. We established inclusion relations between ph-open functions, open functions, and p-open functions, as well as between continuity and ph-o-continuity, confirming in each case that the converse is not true through counterexamples. Furthermore, a new set of ph-separation axioms (T_0 to T_2) was constructed and investigated, which includes their standard counterparts and demonstrates the unidirectional inclusion relationships between them.

Conflict of Interest

The authors declare no conflict of interest.

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