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Effect of End Mass Rotation on the Dynamic Characteristics of a Cantilever Beam Using the Rayleigh-Ritz Principle

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A B S T R A C T

The energy form of the Rayleigh-Ritz Principle was used to investigate the effect of rotating a mass at the free end on the dynamic characteristics of the cantilever beam. The complexity of boundary conditions caused by the end mass was avoided by applying the Rayleigh-Ritz principle. A weak form (energy form) Rayleigh Quotient for the beam with an end mass was derived. The stiffness and mass matrices of the system were obtained using 12 simple fixed-free (without end mass) eigenfunctions as admissible functions. Three different values of the end mass shape factor α (1, 2, and 3×10^{-3}), representing the mass moment of inertia of the end mass about the junction point, were used. The results showed that the influence of α on changing dynamic characteristics was significant in the second mode and above. The observed changes were 1.49%, 3.76% and 7.12% in the second, third, and fourth natural frequencies, respectively. A simple mathematical model was proposed to represent these changes. The end mass and its rotation transform the $N + 1$ mode shape of the cantilever into N-mode shape of a fixed-pinned beam.

1. Introduction

A single flexible manipulator is a crucial component of robotic technology. Since it offers several advantages over traditional rigid manipulators, these include reduced overall mass, lower energy consumption, fewer actuator requirements, and decreased overall cost [1-6]. A single flexible manipulator can be modeled as a fixed-free beam, with one end connected to a motor and the other end carrying a payload of mass M_p, Fig. 1. In Fig.(1) $y(x,t)$ represents the elastic deformation of the beam at x.

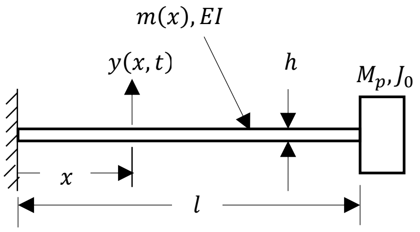

Fig. 1. Cantilever beam with end mass at the free end

The manipulator can be modeled as a uniform, fixed free beam with length l, second moment of area I, and mass per unit length $m(x)$ that carries an end mass M_p. For simplicity, shear deformation and rotary inertia effects are ignored (Euler-Bernoulli beam). To reduce the influence of gravity, the beam was assumed to vibrate in the horizontal plane. Additionally, the elongations of the beam are considered small compared to its transverse vibrations.

Many researchers have focused on studying how the end condition of a fixed-free cantilever beam affected its dynamic characteristics. The impact of the parameter M_p/ml on the maximum dynamic stress for the first mode was presented by [7]. Meanwhile, [8] determines the roots of the frequency equation for a cantilever beam elastically restrained by a linear and torsional spring with an end mass. The regula falsi method was used to solve the frequency equation, taking into account the rotation of the end mass. Additionally, [9] numerically solved the frequency equation for a cantilever beam with a spring mass system attached to the end mass, resulting in extensive numerical data.

More complex boundary conditions can be created when translational and rotational elastic supports are added to the end mass [10]. The exact and analytical expressions for natural frequencies and mode shapes have been derived, and extensive numerical data have been published.

By expressing the boundary conditions in terms of the dynamic stiffness of the attached element (mass, spring, or dashpot), a new method was developed by [11]. This method is used to calculate the natural frequencies and mode shapes for the Euler-Bernoulli beam, regardless of the combination of linear boundary conditions.

The transcendental equation for a clamped-pin-free beam with mass attached at the free end was derived for any pin location by [12]. The comparison between experimental and analytical results showed good agreement in frequency values for the first three modes; however, this agreement deteriorated at higher modes due to the neglect of the rotational inertia of the end mass.

When the beam vibrates, each point on the beam experiences two types of motion: 1)- translational motion $y(x,t)$ and 2)- rotational motion due to changes in the slope $\partial\, y(x,t)/\partial x$ of the point, and the end mass. Some literature implicitly mentions the rotation of the end mas. In this study, the effect of rotation ($\partial\, y(x,t)/\partial x|_{x=l}$) of end mass on the system's natural frequencies and mode shapes was explicitly investigated.

Fig. 2, shows the geometry of the end mass, which was a prism with dimensions ($W \times L \times H$).

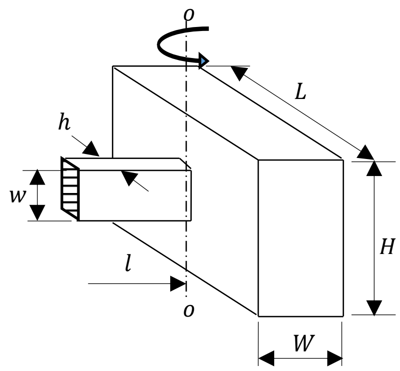

Fig. 2. Dimension of the end mass

2. Rayleigh-Ritz Modelling

To avoid the complexity of boundary conditions of the beam caused by the end mass M_p, energy form of the Rayleigh-Ritz method was used since an admissible (satisfying geometric boundary conditions) function is sufficient to determine the required natural frequencies and normal modes [13].

2.1 Rayleigh Quotient for Cantilever Beam with End Mass

The partial differential equation governing the bending free vibration of a beam is well known as

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right] = m(x)\frac{\partial^2 y(x,t)}{\partial t^2} \qquad (1)$$

In Eq. (1), $m(x)$ is the mass per unit length (kg/m), E is the modulus of elasticity (N/m^2), $I(x)$ is the cross-sectional area moment of inertia (m^4), $EI(x)$ is the flexural rigidity (N $\cdot$ m^2), $y(x,t)$ is the transverse displacement (m), and t is the time (s).

If the beam is considered to have a uniform cross-section and with homogeneous and isotropic material, Eq. (1) reduces to

$$-\left[EI\frac{\partial^4 y(x,t)}{\partial x^4}\right] = m\frac{\partial^2 y(x,t)}{\partial t^2} \qquad (2)$$

The cantilever beam has the following boundary conditions:

1- Deformation

$$y(x,t)|_{x=0} = 0 \qquad (3)$$

2- Rotation

$$\left.\frac{\partial y}{\partial x}\right|_{x=0} = 0 \qquad (4)$$

While the bending moment and shearing force at the free end are as follows:

3- The bending moment

$$\left.EI\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=l} = -J_o\left.\frac{\partial^3 y}{\partial x\partial t^2}\right|_{x=l} \qquad (5)$$

Where: J_o is the mass moment of inertia of the end mass about the o-o axis (Fig. (2)). Note that, $\frac{\partial y}{\partial x} = \theta$ so that the $\frac{\partial^3 y(x,t)}{\partial x\partial t^2} = \frac{\partial^2\theta}{\partial t^2}$ which is the angular acceleration of the end mass.

4- The shearing force

$$\left.EI\frac{\partial^3 y(x,t)}{\partial x^3}\right|_{x=l} = M_p\left.\frac{\partial^2 y}{\partial t^2}\right|_{x=l} \qquad (6)$$

The derivation of the Rayleigh Quotient for the cantilever beam follows [14].

2.1.1 Differential Eigenvalue Problem

The first step in deriving the Rayleigh Quotient is to find the eigenvalue problem associated with the governing equation of the problem and its boundary conditions. To accomplish this, let

$$y(x,t) = Y(x)\cos\omega t \qquad (7)$$

In Eq. (7), the $Y(x)$ is a function of x and ω is the circular frequency. Inserting Eq. (7) into Eq. (2) yields,

$$EI\frac{d^4 Y(x)}{dx^4} = m\omega^2 Y(x) \qquad (8)$$

and the boundary conditions (3), (4), (5), and (6) reduce to

$$Y(x)|_{x=0} = 0 \quad (9) \quad , \quad \left.\frac{dY(x)}{dx}\right|_{x=0} = 0 \quad (10)$$

$$\left.EI\frac{d^2 Y(x)}{dx^2}\right|_{x=l} = J_o\omega\left.\frac{dY(x)}{dx}\right|_{x=l} \qquad (11)$$

and,

$$\left.EI\frac{d^3 Y(x)}{dx^3}\right|_{x=l} = -M_p\omega^2\, Y(x)|_{x=l} \qquad (12)$$

Eq. (8) and the boundary conditions (9), (10), (11) and (12) constitutes the strong form of the eigenvalue problem of the beam with end mass M_p, where $Y(x)$ satisfy every point of Eq. (8) and the boundary conditions (9), (10), (11) and (12).

2.1.2 Weak Form or Energy Form of Rayleigh Quotient

To derive the Rayleigh Quotient for the problem at hand, it is necessary to cast the differential eigenvalue problem in a weak form. This can be achieved by multiplying Eq. (8) by the test function $P(x)$, which act as the role of a weighting function, and by replacing the exact solution $Y(x)$ with the trial function $\phi(x)$, then integrating over the length of the beam

$$\int_0^l EI\,P(x)\frac{d^4\phi}{dx^4}dx = \int_0^l m\,\omega^2 P(x)\,\phi(x)dx \qquad (13)$$

In the Equation (13) the test function $P(x)$ satisfy the boundary conditions at $x = 0$, that are $P(x) = 0$ and $dP/dx = 0$ at $x = 0$, while trail function $\phi(x)$ may be choose admissible functions of the cantilever beam (that satisfy only the geometric boundary condition and are once differentiable). Under these conditions of $P(x)$, integrating the left side of Equation (13) twice by part yield

$$\left.P(x)EI\frac{d^3\phi(x)}{dx^3}\right|_{x=l} - \frac{dP(x)}{dx}EI\left.\frac{d^2\phi(x)}{dx^2}\right|_{x=l}$$
$$+\int_0^l EI\frac{d^4\phi}{dx^4}\frac{d^2P(x)}{dx^2}dx$$
$$=\int_0^l m\,\omega^2 P(x)\,\phi(x)dx \qquad (14)$$

Equation (14) illustrates the weak form of the eigenvalue problem for the cantilever beam with an end mass. If the test function $P(x)$ is chosen as the trail function $\phi(x)$, and considering the boundary conditions (9), (10), (11), and (12), the Energy form of the Rayleigh quotient for this problem is

$$R(\omega) = \omega^2 = \frac{\frac{1}{2}\int_0^l \left[EI(x)\frac{d^2\phi(x)}{dx^2}\right]^2 dx}{\frac{1}{2}\int_0^l m(x)\phi^2(x)\,dx + \frac{1}{2}M_p\phi^2(l) + \frac{1}{2}J_o\left(\left.\frac{d\phi}{dx}\right|_{x=l}\right)^2} \quad (15)$$

The numerator of Eq. (15) indicates the maximum elastic potential energy of the beam, while the three terms in the denominator are

Term (1) represents the kinetic energy of the beam.
Term (2) refers to the kinetic energy resulting from the translational motion of the end mass.
Term (3) refers to kinetic energy resulting from the rotational motion of the end mass. (objective of this study)

The Rayleigh-Ritz method is performed by assuming $\phi(x)$ in the form of a linear combination of the trial functions $\phi_i(x)$

$$\phi^{(n)}(x) = \sum_{i=1}^{n} a_i\phi_i(x) \quad (16)$$

Where $\phi_i(x)$ represent independent trial functions and a_i, are undetermined coefficients, while n is the number of terms of the trial function $\phi_i(x)$.

For simplicity, the independent trial functions $\phi_i(x)$ of the fixed-free (without end mass) beam were chosen, which gives by

$$\phi_r(x) = (\sin\beta_r x - \sinh\beta_r x) - \frac{\sin\beta_r l + \sinh\beta_r l}{\cos\beta_r l + \cosh\beta_r l}(\cos\beta_r x + \cosh\beta_r x) \quad (17)$$

where β_r are roots of the frequency equations

$$\cos\beta_r l \cosh\beta_r l = -1 \quad (18)$$

The detailed derivation of Eq.(17) is provided in the Appendix. Maple software was used to find the roots of Eq. (18). The first 12 values of β_r are listed in Table (1). By introducing the specific $\phi^{(n)}(x)$ into Eq. (15) and making Rayleigh's quotient stationary, the stiffness and mass matrices become.

$$k_{ij} = k_{ji} = \int_0^l EI(x)\frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}dx \ , \ i,j = 1,2,\cdots,n \quad (19)$$

$$m_{ij} = m_{ji} = \int_0^l m(x)\,\phi_i(x)\phi_j(x)dx + M_p\phi_i(l)\phi_j(l) + J_o\phi_i'(l)\phi_j'(l) \quad i,j = 1,2,\cdots,n \quad (20)$$

Table 1. The roots β_r of Eq. (18)

r	1	2	3	4	5	6
β_r	1.86	4.67	7.87	11.0	14.17	17.32
r	7	8	9	10	11	12
β_r	20.5	23.61	26.7	29.9	33.06	36.2

The main parameters of the beam are identified in Table 2.

Table 2. Parameters of the cantilever beam

Material	Stainless steel
Thickness (h)	$3.3 \times 10^{-3}\ m$
Width (w)	$25 \times 10^{-3}\ m$
Length (l)	$0.5\ m$
Area moment of inertia (I)	$7.487 \times 10^{-11} m^4$
Bending stiffness (EI)	$14.974\ N.m^2$

3. Results and Discussion

A MATLAB program was developed to compute the stiffness matrix (Eq. 19) and mass matrix (Eq. 20). Natural frequencies and mode shapes were found using MATLAB's eigenvalue solver. Twelve terms ($n = 12$) were used, with M_p/ml set to 1; the constant M_p remained the same for all J_o values examined in this study.

Three constraints were applied when determining the dimensions of the end mass, to achieve different values of the mass moment of inertia J_o about the rotation axis o-o. First, the width W (Fig. 2) was held constant, which kept the centre of gravity at the same position for all shapes. Second, the mass M_p was the same for all shapes and satisfied the ratio $M_p/ml = 1$. Third, a specific value of α was achieved by changing the other two dimensions (L and H). Where $\alpha = J_o/l^2/ml$ is the dimensionless end mass shape factor. Note that $\alpha = 0$ means that the rotation of M_p about o-o axis is ignored.

Table 3. Dimensions of the end mass.

α $\times 10^{-3}$	J_o (kg. m^2) $\times 10^{-3}$	L (mm)	H (mm)	W (mm)
0	ignored	50	50	13.2
1	8.173	48	52	13.2
2	16.346	72.8	34.3	13.2
3	24.52	91.1	27.4	13.2

The dimensions of the end mass that satisfy the specific values of α, namely $(1,2,3)\times10^{-3}$, are shown in Table 3.

To verify the reliability of the computer program, the results were compared with the exact solution for $\alpha = 0$, as given in Ref [14]. Excellent results were obtained, as shown in Table 4.

3.1. Natural frequencies

The first four natural frequencies for three different values of α are tabulated in Table 5.
To visualize the changes that appear in the natural frequencies with α Fig. 3, they were plotted.

Table 4. Exact and present first three natural frequencies for $\alpha = 0$.

Mode No.	Rayleigh Ritz (Hz)	Exact (Hz) (Ref. 14)
ω_1	4.747	4.744
ω_2	49.53	49.507
ω_3	155.14	155.057
ω_4	320.4	318.8

Table 5. Effect of α on first three natural frequencies of end mass cantilever beam.

α $\times 10^{-3}$	First three natural frequencies (Hz)			
	ω_1	ω_2	ω_3	ω_4
0	4.747	49.53	155.14	320.38
1	4.746	49.28	153.06	312.41
2	4.745	49.04	151.0	304.74
3	4.744	48.79	149.0	297.6

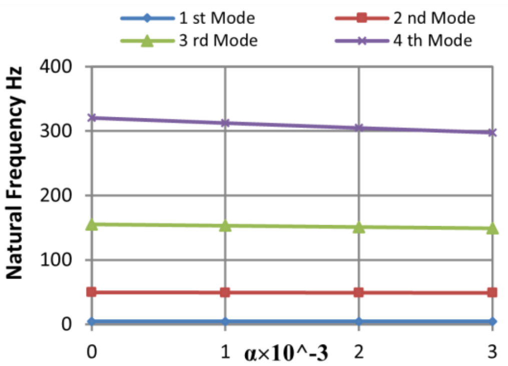

Fig. 3. Effect of α on the first four natural frequency

From Fig. 3, it can be seen that the effect of α on the first natural frequency is almost negligible, while the reduction in the fourth natural frequency is clear as α increases. When α increases from 0 to 3×10^{-3}, the percentage reductions in natural frequencies relative to $\alpha = 0$ are as follows: first 0%, second 1.49%, third 3.76% and fourth 7.12%. This is due to the dependence of the moment produced by J_o on the angular acceleration of J_o.

A simple empirical equation for these reductions relating the mode number, N, and the dimensionless end mass shape factor α was suggested as

$$\omega_n = [\omega_{N0} + (-0.13671N^3 + 0.58184N^2 - 0.86966N + 0.42453)\times\alpha] \times \sqrt{\frac{EI}{ml}} \qquad (21)$$

Note that Eq. (21) applies only when for $M_p/ml = 1$.

Equation (21) can be used to estimate the natural frequencies in rad/sec for a uniform cantilever beam carrying end mass if ω_{N0} (N-mode natural frequency in rad/sec of the beam for $\alpha = 0$) is known. Table (6) shows excellent agreement between the results obtained by the empirical equation and those obtained by the Rayleigh Ritz principle.

Table 6. Comparison between Rayleigh Ritz and empirical Eq. (21)

Mode No.	α $\times 10^{-3}$	R.R	Emp.
ω_1 (Hz)	0	4.747	4.747
	1	4.746	4.747
	2	4.745	4.747
	3	4.744	4.747
ω_2 (Hz)	0	49.53	49.53
	1	49.28	49.28
	2	49.04	49.036
	3	48.79	48.79
ω_3 (Hz)	0	155.14	155.14
	1	153.06	153.19
	2	151.0	151.24
	3	149	149.3
ω_4 (Hz)	0	320.38	320.38
	1	312.41	312.78
	2	304.74	305.18
	3	297.6	297.57

3.2 Mode Shapes

The accurate mode shapes is crucial to determine response of the beam to external excitation and in the sensors placement [15].

The first four normalized to 1 modes depicted in Figs. 4-7. The effect of α can be seen from second mode and above, and clearly can be seen in fourth mode. For comparison first three normalized to 1 modes of fixed-pinned (without end mass) were plotted, Fig. 8.

The main observations are that the end mass M_p and increasing α tend the N + 1 mode of cantilever beam with end mass approaches N mode of fixed-pinned beam. The rigidity of M_p and the contribution of α are justify it.

4. Conclusion

In this study, the energy form Rayleigh-Ritz method was used to investigate the effect of the rotation end mass represented by α on the dynamic characteristics of the cantilever beam. The main conclusions from this study can be summarized as follows:

1- A good result was obtained when using 12 admissible functions in the Rayleigh-Ritz principle.

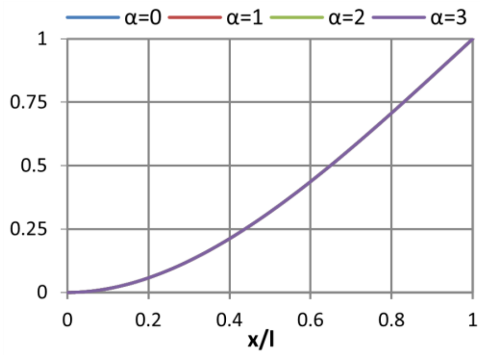

Fig. 4. Negligible Effect of α on fist mode.

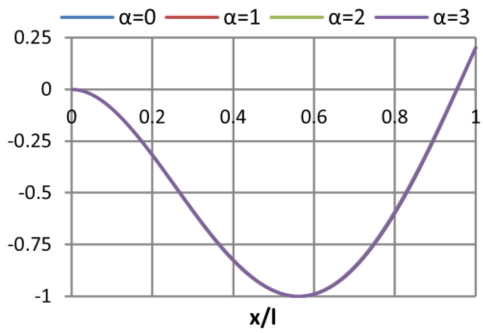

Fig. 5. Effect of α to 2 nd mode.

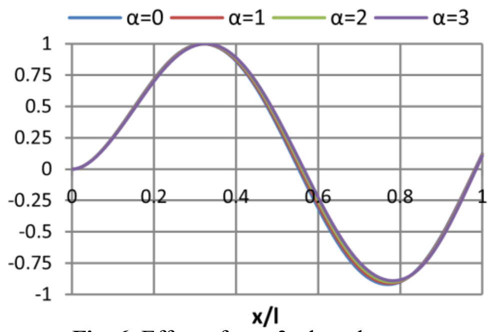

Fig. 6. Effect of α to 3 rd mode.

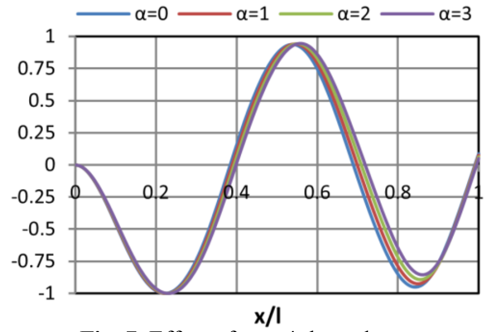

Fig. 7. Effect of α to 4 th mode.

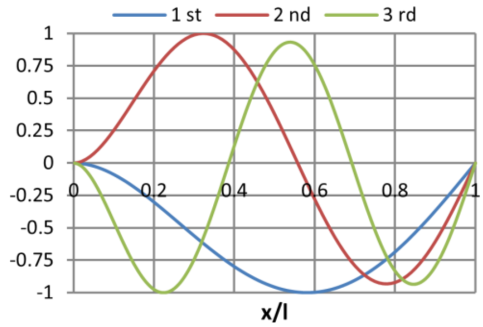

Fig. 8. The first three modes of fixed-pin beam

2- The effect of α on both natural frequencies and mode shapes has been seen on the second mode and above

3- The percentage of change in natural frequencies increases with increasing α.

4- The complexity in the boundary condition can be avoided by using the energy form Rayleigh-Ritz principle.

5- The simple mathematical model suggested to estimate the reduction in natural frequencies with increasing α.

6- The end mass and its rotation convert N + 1 mode shape of the cantilever to the N-mode shape of a fixed-pinned beam without end mass.

References

[1] H.-J. Yang and M. Tan, "Sliding mode control for flexible-link manipulators based on adaptive neural networks," *International Journal of Automation and Computing,* vol. 15, pp. 239-248, 2018.

[2] C. Sun, W. He, and J. Hong, "Neural network control of a flexible robotic manipulator using the lumped spring-mass model," *IEEE Transactions on Systems, Man, and Cybernetics: Systems,* vol. 47, pp. 1863-1874, 2016.

[3] H. Gao and W. He, "Fuzzy control of a single-link flexible robotic manipulator using assumed mode method," in *2016 31st Youth Academic Annual Conference*

of Chinese Association of Automation (YAC), 2016, pp. 201-206.

[4] M. A. Ahmad, M. A. Zawawi, and Z. Mohamed, "Effect of Beam's Length on the Dynamic Modelling of Flexible Manipulator System," in *2009 11th International Conference on Computer Modelling and Simulation*, 2009, pp. 356-361.

[5] M. Khairudin, "Dynamic modelling of a flexible link manipulator robot using AMM," *Telkomnika,* vol. 6, p. 185, 2008.

[6] M. Galicki , "Kinematically Optimal Robust Control of Redundant Manipulators", Int. J. of Applied Mechanics and Engineering., vol. 22, pp. 839-865, 2017.

[7] P. Laura, J. Pombo, and E. Susemihl, "A note on the vibrations of a clamped-free beam with a mass at the free end," *Journal of Sound and Vibration,* vol. 37, pp. 161-168, 1974.

[8] M. Gürgöze and H. Batan, "A note on the vibrations of a restrained cantilever beam carrying a heavy tip body," *Journal of Sound Vibration,* vol. 106, pp. 533-536, 1986.

[9] M. Gürgöze, "On the eigenfrequencies of a cantilever beam with attached tip mass and a spring-mass system," *Journal of Sound and Vibration,* vol. 190, pp. 149-162, 1996.

[10] D. Zhou, "The vibrations of a cantilever beam carrying a heavy tip mass with elastic supports," *Journal of sound and vibration,* vol. 206, pp. 275-279, 1997.

[11] P. J. P. Gonçalves, M. J. Brennan, A. Peplow, and B. Tang, "Calculation of the natural frequencies and mode shapes of a Euler–Bernoulli beam which has any combination of linear boundary conditions," *Journal of Vibration and Control,* vol. 25, pp. 2473-2479, 2019.

[12] J. Hong, J. Dodson, S. Laflamme, and A. Downey, "Transverse vibration of clamped-pinned-free beam with mass at free end," *Applied Sciences,* vol. 9, p. 2996, 2019.

[13] Y. Kumar, "The Rayleigh–Ritz method for linear dynamic, static and buckling behavior of beams, shells and plates: A literature review," *Journal of Vibration and Control,* vol. 24, pp. 1205-1227, 2018.

[14] L. Meirovitch, "*Fundamentals of vibrations*," McGraw-Hill International Edition, 2001.

[15] K. Lim and W. Gawronski, "Actuator and sensor placement for control of flexible structures," *Control and Dynamics Systems,* pp. 109-152, 2012.

Appendix

The partial differential equation for the bending free vibration of a beam in the form

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right] = m(x)\frac{\partial^2 y(x,t)}{\partial t^2} \tag{A1}$$

To drive the eigenvalue of the problem (fixed-free beam without end mass) the separation of variables method, was used, let

$$y(x,t) = Y(x)T(t) \tag{A2}$$

Inserting Eq. (A2) and its derivatives with respect to x and t, Eq. (A1) is reduced to

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y(x)}{dx^2}\right]T(t) + m(x)Y(x)\ddot{T} = 0 \tag{A3}$$

if $I(x) = \text{constant} = I$, and by separating the variables, the Eq. (A3) will be

$$\frac{EI}{mY(x)}\frac{d^4Y(x)}{dx^4} = -\frac{\ddot{T}}{T} = \omega^2$$

$$\frac{d^4Y(x)}{dx^4} - \beta^4 Y(x) = 0 \tag{A4}$$

Where,

$$\beta^4 = \frac{m(x)\omega^2}{EI}$$

The solution of Eq. (A4) is

$$Y(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x \tag{A5}$$

By considering the boundary conditions of a fixed-free cantilever beam, namely

$$Y(x)|_{x=0} = \left.\frac{dY(x)}{dx}\right|_{x=0} = 0 \tag{A6}$$

Hence, Eq. (A5) reduced to

$$Y(x) = A(\sin\beta x - \sinh\beta x) + B(\cos\beta x - \cosh\beta x) \tag{A7}$$

Then, the second and third derivatives of (A7) are

$$\frac{d^2Y(x)}{dx^2} = -\beta^2[A(\sin\beta x + \sinh\beta x) + B(\cos\beta x + \cosh\beta x)] \tag{A8}$$

$$\frac{d^3Y(x)}{dx^3} = -\beta^3[A(\cos\beta x + \cosh\beta x) - B(\sin\beta x - \sinh\beta x)] \tag{A9}$$

At the free end, the boundary condition at $x = l$ are

$$EI\left.\frac{d^2Y(x)}{dx^2}\right|_{x=l} = EI\left.\frac{d^3Y(x)}{dx^3}\right|_{x=l} = 0 \tag{A10}$$

Now, enforcing the first condition of (A10) to (A8), the relation between constants A and B will be

$$B = -\frac{\sin\beta l + \sinh\beta l}{\cos\beta l + \cosh\beta l}A \tag{A11}$$

Inserting Eq.(A11) into Eq. (A9) then, applying second condition of (A10)

$$0 = A[\cos\beta l + \cosh\beta l + \frac{\sin\beta l + \sinh\beta l}{\cos\beta l + \cosh\beta l}(\sin\beta l - \sinh\beta l)\}$$

Since $A \neq 0$ and using trigonometric and hyperbolic identities

$$\sin^2\beta l + \cos^2\beta l = 1 \quad \text{and} \quad \cosh^2\beta l - \sinh^2\beta l = 1$$

The frequency equation of a fixed-free cantilever beam becomes

$$\cos\beta_r l\cosh\beta_r l = -1 \tag{A12}$$

Where r is the number of roots. Inserting Eq. (A11) into (A7) to obtain the r-th Eigenfunction of a fixed-free cantilever beam (without end mass), which is used to investigate the effect of rotation end mass by the Rayleigh-Ritz method.

$$\phi_n(x) = A_r\left[\sin\beta_r x - \sinh\beta_r x - \frac{\sin\beta_r l + \sinh\beta_r l}{\cos\beta_r l + \cosh\beta_r l}(\cos\beta_r x - \cosh\beta_r x)\right] \tag{A13}$$